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On Weak Convergence and Metrization Issues in Separable Hilbert Spaces

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ABSTRACT

Weak convergence is a fundamental concept in infinite-dimensional analysis because it provides a compactness framework when norm convergence is unavailable. In separable Hilbert spaces, a basic dichotomy appears: the weak topology is not metrizable on the whole infinite-dimensional space, while it becomes metrizable on bounded subsets. This distinction plays an important role in functional analysis and vibrational methods. This paper adopts a rigorous theoretical-analytical approach within the setting of separable Hilbert spaces. The study uses orthonormal bases, bounded sequences, weak topology, and compactness arguments to characterize weak convergence and to construct an explicit metric on closed balls. Classical tools from Hilbert space theory, weak compactness, and lower semicontinuity are employed throughout. The paper proves that, for bounded sequences, weak convergence is equivalent to coordinatewise convergence with respect to a fixed orthonormal basis. It also constructs an explicit metric that induces the weak topology on bounded subsets and weakly compact sets. In contrast, it is shown that the weak topology on the whole infinite-dimensional Hilbert space is never metrizable. The results clarify that boundedness and separability are the key conditions behind the metrization of weak topology in Hilbert spaces. This dichotomy explains weak compactness, extraction of weakly convergent subsequences, and weak lower semicontinuity of the norm. It also provides a useful framework for direct methods in variational analysis and related applications.

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1.INTRODUCTION

The advantage of weak convergence has been taking a privileged role in functional analysis as it holds strong convergence in many cases. In partial differential equations, calculus of variations, convex optimization, and fixed-point theory a common phenomenon is to have an application of bounded sequences and no norm convergent sub-sequences. This weakness is addressed by the weak topology that gives a framework whereby compactness phenomena can still be seen in the context of bounded families. Classical sources like Brezis, Conway, Megginson and Robinson elaborate on this point of view in finer detail and repeat the point that weak compactness is a pillar of infinite-dimensional analysis [1], [2], [3], [4] This is further demonstrated by the compactness theorem of Alaoglu and weak compactness theory introduced by Eberlein which demonstrates the need of weak topology in modern analysis [5], [6], [7] Weak convergence is particularly clear in Hilbert spaces, since by the Riesz representation program, the space itself is also the continuous dual. The weak topology, even in this pleasant environment, however, has a small topological dichotomy. On the one hand, in case of the Hilbert space being of an infinite form, the weak topology of the entire space is not metrizable. Conversely, in case the space is separable, then the weak topology is also metrizable on all norm-bounded sets. The consequences of this simple appearance of contrast are analytical. It characterizes the reason bounded sequences have weakly convergent subsequences, the reason why coordinate tests using orthonormal bases can only be made to work under boundedness conditions and the reason why many variational arguments will always occur within bounded weakly compact subsets of the ambient space [1], [8] This paper is aimed at providing a concise and self-contained discussion of such matters regarding separable Hilbert spaces. To be more exact, the following are our goals.

1. To demonstrate that weak convergence to bounded sequences is co-ordinatewise convergence relative to an orthonormal basis fixed.

2. In order to build an explicit measure inducing the weak topology on any closed ball.

3. To establish it is not possible to metrize the global weak topology on an infinite-dimensional Hilbert space.

4. To infer implications on compactness, extraction of sub-sequences, weak lower semicontinuity and variational existence theory.

The results are classical in their spirit, even the presentation, but it is framed using the metric point of view based on the basis, which reveals the role of separability and limitedness in particular. Not only is this point of view conceptually convenient; it is also technically efficient in those cases when one is interested in moving between finite-dimensional approximations to weak limits in the full Hilbert space.

2. Preliminaries and notation

Let H be a real or complex separable Hilbert space. We denote its inner product by $\langle\cdot,\cdot\rangle$ and its norm by $\| x \|=\sqrt{\langle x, x\rangle}$. Weak convergence in H is denoted by $x_n \rightharpoonup x$ and means that

$$\langle x_n, y\rangle \to \langle x, y\rangle \quad \text{for every } y \in H.$$

Since H is separable, there exists a fixed orthonormal basis $\{e_k\}_{k\geq 1}$. For each $N \in \mathbb{N}$, let $P_N: H \to H$ be the orthogonal projection onto

$$H_N := \operatorname{span}\{e_1, \dots, e_N\}.$$

Then $P_N z \to z$ strongly for every $z \in H$.

Proposition 2.1. If $x_n \rightharpoonup x$ in H, then $\sup_n \| x_n \| < \infty$.

Proof. For each $n \in \mathbb{N}$, define the continuous linear functional $T_n: H \to \mathbb{K}$ by

$$T_n(y) := \langle x_n, y\rangle,$$

where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Since $x_n \rightharpoonup x$, for every fixed $y \in H$ the scalar sequence $T_n(y)$ converges to $\langle x, y\rangle$ and is therefore bounded. By the Banach–Steinhaus theorem, $\sup_n \| T_n \| < \infty$. Using the Riesz representation theorem, $\| T_n \| = \| x_n \|$ for every n, and the conclusion follows [1], [2]

Proposition 2.1 is elementary but decisive: weak convergence always carries an implicit boundedness statement. Much of the analysis below consists in identifying precisely where this hidden boundedness is needed.

3. Weak convergence through orthonormal coordinates

Theorem 3.1. Let $\{e_k\}_{k\geq 1}$ be an orthonormal basis of H. Let (x_n) be a sequence in H and let $x \in H$. Then the following are equivalent:

(a) $x_n \rightharpoonup x$ in H.

(b) The sequence (x_n) is bounded and

$$\langle x_n, e_k\rangle \to \langle x, e_k\rangle \qquad \text{for every } k \in \mathbb{N}.$$

Proof. The implication (a)⇒(b) follows immediately from Proposition 2.1, because each map $z \mapsto \langle z, e_k \rangle$ is a continuous linear functional.

Now assume (b). Set

$$M := \sup_n \| x_n \| + \| x \| < \infty.$$

Let $z \in H$ and $\varepsilon > 0$ be arbitrary. Since $P_N z \to z$ in norm, we may choose N so large that

$$2M \| z - P_N z \| < \frac{\varepsilon}{2}.$$

Because $P_N z \in H_N = \operatorname{span}\{e_1, \dots, e_N\}$, there exist fixed scalars $a_1, \dots, a_N$ such that

$$P_N z = \sum_{k=1}^{N} a_k \, e_k.$$

Hence

$$\langle x_n - x, P_N z \rangle = \sum_{k=1}^{N} c_k \, \langle x_n - x, e_k \rangle,$$

for suitable fixed scalars $c_1, \dots, c_N$ depending only on z and the inner-product convention. By the assumed coordinate convergence, there exists n_0 such that for all $n \geq n_0$,

$$|\langle x_n - x, P_N z \rangle| < \frac{\varepsilon}{2}.$$

For such n we have

$$\begin{aligned} |\langle x_n - x, z \rangle| &\leq |\langle x_n - x, P_N z \rangle| + |\langle x_n - x, z - P_N z \rangle| \\ &\leq \frac{\varepsilon}{2} + \| x_n - x \| \, \| z - P_N z \| \\ &\leq \frac{\varepsilon}{2} + 2M \| z - P_N z \| < \varepsilon. \end{aligned}$$

Therefore $\langle x_n, z \rangle \to \langle x, z \rangle$ for every $z \in H$, i.e. $x_n \rightharpoonup x$.

Example 3.2 (Why boundedness is indispensable). Consider $H = \ell^2$ with its canonical orthonormal basis (e_n). Define $x_n := n e_n$. Then for each fixed $k \in \mathbb{N}$ we have

$$\langle x_n, e_k \rangle \to 0,$$

because the kth coordinate vanishes for all sufficiently large n. Nevertheless (x_n) does *not* converge weakly to 0. Indeed, if

$$y := \sum_{n=1}^{\infty} \frac{1}{n} e_n \in \ell^2,$$

then

$$\langle x_n, y \rangle = 1 \qquad \text{for every } n.$$

Thus coordinatewise convergence on a basis does not imply weak convergence without a boundedness assumption.

4. Metrization on bounded subsets

For $R > 0$, let

$$B_R(0) := \{x \in H : \| x \| \leq R\}.$$

Define, for $x, y \in B_R(0)$,

$$d_R(x, y) := \sum_{k=1}^{\infty} 2^{-k} \min\{1, |\langle x - y, e_k \rangle|\}.$$

The series converges absolutely because each term is bounded by 2^{-k}.

Theorem 4.1. For every $R > 0$, the metric d_R induces the weak topology on $B_R(0)$.

Proof. We first note that d_R is indeed a metric. Symmetry and definiteness are immediate. For the triangle inequality, observe that for $a, b \geq 0$,

$$\min\{1, a + b\} \leq \min\{1, a\} + \min\{1, b\}.$$

Applying this coordinatewise and summing with the weights 2^{-k} gives the claim.

Assume first that $x_n \rightharpoonup x$ in $B_R(0)$. Then, for each fixed k, we have

$$\langle x_n - x, e_k \rangle \to 0.$$

Hence every summand tends to 0, and dominated convergence yields $d_R(x_n, x) \to 0$.

Conversely, suppose that $d_R(x_n, x) \to 0$. Since every term is nonnegative, we must have

$$\min\{1, |\langle x_n - x, e_k \rangle|\} \to 0 \qquad \text{for every } k \in \mathbb{N}.$$

Therefore $\langle x_n, e_k \rangle \to \langle x, e_k \rangle$ for every k. Because all points lie in $B_R(0)$, the sequence (x_n) is bounded. By Theorem 3.1, it follows that $x_n \rightharpoonup x$.

Remark 4.2. The metric d_R depends on the chosen orthonormal basis, but the topology it induces does not. Any orthonormal basis of H gives a metricization of the weak topology on $B_R(0)$.

Corollary 4.3. Every weakly compact subset of a separable Hilbert space is metrizable in its weak topology.

Proof. Let $K \subset H$ be weakly compact. Weakly compact subsets of Banach spaces are norm bounded [3], [9] Hence $K \subset B_R(0)$ for some $R > 0$. The weak topology on K is the subspace topology inherited from the weak topology on $B_R(0)$, and by Theorem 4.1 that topology is induced by the restriction of d_R to K.

Example 4.4 (The canonical basis in ℓ^2). Let $H = \ell^2$ and let (e_n) be the canonical orthonormal basis. Then $e_n \rightharpoonup 0$ but e_n does not converge to 0 in norm. Indeed,

$$\langle e_n, y \rangle = y_n \to 0 \qquad \text{for every } y = (y_n) \in \ell^2,$$

so $e_n \rightharpoonup 0$, while $\| e_n \| = 1$ for all n. In the metric d_1 on the unit ball, one obtains explicitly

$$d_1(e_n, 0) = 2^{-n} \to 0.$$

This illustrates both the weakness of weak convergence and the concreteness of the metricization on bounded sets.

5. Why the global weak topology is not metrizable

Theorem 5.1. Let H be an infinite-dimensional Hilbert space. Then the weak topology $\sigma(H, H)$ is not metrizable on the whole space.

Proof. Assume, for contradiction, that $\sigma(H, H)$ is metrizable. Since it is a locally convex topology, there exists a countable family of weakly continuous seminorms generating it. Because weakly continuous seminorms arise from continuous linear functionals, we may assume that the topology is generated by seminorms of the form

$$p_n(x) := |\langle x, y_n \rangle|, \qquad n \in \mathbb{N},$$

for some sequence $(y_n) \subset H$.

Let $f \in H$ be arbitrary, and consider the weakly continuous linear functional

$$\varphi_f(x) := \langle x, f \rangle.$$

Since φ_f is continuous at 0 for the topology generated by (p_n), there exist indices $n_1, \ldots, n_m$ and $\varepsilon > 0$ such that

$$\max_{1\le j\le m} p_{n_j}(x) < \varepsilon \quad \Rightarrow \quad \left|\varphi_f(x)\right| < 1.$$

By homogeneity, for arbitrary $x \in H$ this implies

$$\left|\varphi_f(x)\right| \le \varepsilon^{-1} \sum_{j=1}^{m} p_{n_j}(x).$$

Therefore, if $x \in \bigcap_{j=1}^{m} \ker p_{n_j}$, then $\varphi_f(x) = 0$. Equivalently,

$$\bigcap_{j=1}^{m} \ker\langle \cdot, y_{n_j}\rangle \subset \ker\langle \cdot, f\rangle.$$

Taking orthogonal complements yields

$$f \in \operatorname{span}\{y_{n_1}, \dots, y_{n_m}\}.$$

Since $f \in H$ was arbitrary, we conclude that

$$H = \operatorname{span}\{y_n : n \in \mathbb{N}\} = \bigcup_{m=1}^{\infty} \operatorname{span}\{y_1, \dots, y_m\}.$$

Each finite-dimensional subspace $\operatorname{span}\{y_1, \dots, y_m\}$ is closed and has empty interior in the norm topology of the infinite-dimensional Banach space H. The Baire category theorem then implies that H cannot be a countable union of such sets. This contradiction proves that $\sigma(H, H)$ is not metrizable.

Remark 5.2. Theorem 5.1 and Theorem 4.1 together capture the central dichotomy of weak topology in separable Hilbert spaces: separability alone is insufficient for global metrizability, whereas boundedness restores metrizability through countably many coordinates.

6. Compactness and stability consequences

Proposition 6.1. Every bounded sequence in a separable Hilbert space has a weakly convergent subsequence.

Proof. Let (x_n) be bounded. Then $(x_n) \subset B_R(0)$ for some $R > 0$. Since Hilbert spaces are reflexive, the closed ball $B_R(0)$ is weakly compact by the Banach–Alaoglu theorem together with reflexivity[1], [5], [8]. By Theorem 4.1, the weak topology on $B_R(0)$ is metrizable; hence $B_R(0)$ is a compact metric space in its weak topology. Every sequence in a compact metric space admits a convergent subsequence, so some subsequence of (x_n) converges weakly in H.

Remark 6.2. Proposition 6.1 may also be viewed through the Eberlein–Šmulian theorem, which identifies weak compactness and weak sequential compactness in Banach spaces [6], [7]In separable Hilbert spaces, however, the metricization from Theorem 4.1 yields a particularly direct proof.

Proposition 6.3 (Weak lower semicontinuity of the norm). If $x_n \rightharpoonup x$ in H, then

$$\| x \| \leq \liminf_{n\to\infty} \| x_n \|.$$

Proof. If $x = 0$, there is nothing to prove. Assume $x \neq 0$. Since $x_n \rightharpoonup x$, we have

$$\langle x_n, x\rangle \to \langle x, x\rangle = \| x \|^2.$$

By the Cauchy–Schwarz inequality,

$$|\langle x_n, x\rangle| \leq \| x_n \| \| x \|.$$

Passing to the limit inferior gives

$$\| x \|^2 \leq \left(\liminf_{n\to\infty} \| x_n \|\right) \| x \|,$$

and division by $\| x \|$ yields the claim.

Proposition 6.4 (Radon–Riesz principle). Let $x_n \rightharpoonup x$ in H. If, in addition,

$$\| x_n \| \to \| x \|,$$

then $x_n \to x$ strongly in H.

Proof. Using weak convergence with the test vector x, we obtain

$$\langle x_n, x\rangle \to \langle x, x\rangle = \| x \|^2.$$

Hence

$$\begin{aligned} \| x_n - x \|^2 &= \| x_n \|^2 + \| x \|^2 - 2\mathrm{Re}\langle x_n, x\rangle \\ &\to \| x \|^2 + \| x \|^2 - 2 \| x \|^2 = 0. \end{aligned}$$

Therefore $x_n \to x$ in norm.

7. A variational application

Theorem 7.1 (Direct method in a separable Hilbert space). Let $F: H \to (-\infty, +\infty]$ be a proper functional. Assume that:

(i) F is coercive, i.e. $F(x) \to +\infty$ whenever $\| x \| \to \infty$;

(ii) F is weakly lower semicontinuous, i.e. $x_n \rightharpoonup x$ implies

$$F(x) \leq \liminf_{n\to\infty} F(x_n).$$

Then F attains its infimum on H.

Proof. Let

$$\alpha := \inf_{x\in H} F(x).$$

Choose a minimizing sequence (x_n) such that $F(x_n) \to \alpha$. By coercivity, (x_n) is bounded. Proposition 6.1 provides a subsequence, still denoted (x_n), and a point $\bar{x} \in H$ such that $x_n \rightharpoonup \bar{x}$. Weak lower semicontinuity then yields

$$F(\bar{x}) \leq \liminf_{n\to\infty} F(x_n) = \alpha.$$

Since α is the infimum, we also have $\alpha \leq F(\bar{x})$. Therefore $F(\bar{x}) = \alpha$, so $\bar{x}$ is a minimizer.

Remark 7.2. In Hilbert spaces, many functionals of practical interest are weakly lower semicontinuous because they are convex and lower semicontinuous in the norm topology [1], [10]. Thus Theorem 7.1 applies broadly in variational problems, inverse problems, and monotone operator theory.

Example 7.3 (Tikhonov-type regularization). Let G be another Hilbert space, let $A: H \to G$ be bounded and linear, let $f \in G$, and let $\lambda > 0$. Consider

$$J(x) := \frac{1}{2} \| Ax - f \|_G^2 + \frac{\lambda}{2} \| x \|_H^2.$$

Then J is proper, convex, and coercive. Moreover, if $x_n \rightharpoonup x$ in H, then $Ax_n \rightharpoonup Ax$ in G, and the norm-square is weakly lower semicontinuous in a Hilbert space. Hence J is weakly lower semicontinuous and Theorem 7.1 guarantees the existence of at least one minimizer. This is a basic model for regularized least-squares problems in inverse theory.

8. Conclusion

It is a very elementary but useful principle that the weak topology of a separable Hilbert space is the one that in turn the nonmetrizability in the intrinsic sense of an oftentimes and oftentimes infinite-dimensional topology becomes such that the latter can be described using a finite number of coordinates and is inherently endowed with a metrical measure. This principle has been made specific in this paper. We have established that the equivalence between weak convergence to bounded sequences and convergence of the coefficients of an orthonormal basis were established, utilized this equivalence to metricize the weak topology on closed balls, and went on to establish that the global weak topology was nonmetrizable in all infinite-dimensional Hilbert spaces. In this sense, weak compactness, extraction of sub-sequences, lower semicontinuity and conservative variational procedures can be seen as obvious derivatives of the bounded/unbounded dichotomy. A number of directions are the logical extensions of the current discussion. A person can replace weak topology by weak-star topology on duals, study the conditions under which metrizability is needed in a nonseparable context or point to the basis-driven metric approach as the basis of probabilistic questions about Hilbert-valued random variables, and convergence in a probability distribution. In its classical form, nevertheless weak convergence in Hilbert spaces is one of the most helpful methods of connecting geometry with topology with analysis.

References

[1] H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*. New York, NY: Springer New York, 2011. doi: 10.1007/978-0-387-70914-7.

[2] J. B. Conway, *A Course in Functional Analysis*, vol. 96. New York, NY: Springer New York, 2007. doi: 10.1007/978-1-4757-4383-8.

[3] R. E. Megginson, *An Introduction to Banach Space Theory*, vol. 183. New York, NY: Springer New York, 1998. doi: 10.1007/978-1-4612-0603-3.

[4] J. C. Robinson, *An Introduction to Functional Analysis*. Cambridge University Press, 2020. doi: 10.1017/9781139030267.

[5] L. Alaoglu, “Weak Topologies of Normed Linear Spaces,” *The Annals of Mathematics*, vol. 41, no. 1, p. 252, Jan. 1940, doi: 10.2307/1968829.

[6] W. F. Eberlein, “Weak Compactness in Banach Spaces,” *Proceedings of the National Academy of Sciences*, vol. 33, no. 3, pp. 51–53, Mar. 1947, doi: 10.1073/pnas.33.3.51.

[7] S. Baratella and S.-A. Ng, “A nonstandard proof of the Eberlein-Smulian theorem,” *Proceedings of the American Mathematical Society*, vol. 131, no. 10, pp. 3177–3180, Jan. 2003, doi: 10.1090/S0002-9939-03-06894-1.

[8] K. Yosida, *Functional Analysis*, vol. 123. Berlin, Heidelberg: Springer Berlin Heidelberg, 1995. doi: 10.1007/978-3-642-61859-8.

[9] C. D. Aliprantis and K. C. Border, *Infinite Dimensional Analysis*. Berlin, Heidelberg: Springer Berlin Heidelberg, 1999. doi: 10.1007/978-3-662-03961-8.

[10] H. H. Bauschke and P. L. Combettes, *Convex Analysis and Monotone Operator Theory in Hilbert Spaces*. Cham: Springer International Publishing, 2017. doi: 10.1007/978-3-319-48311-5

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