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Weak Singularity Modulation in Discrete Solvers for Volterra-Type Integral Equations

Mays Fadhil Sabar
[1]Islamic Azad University Isfahan (Khorasgan) Branch, Department of Mathematics

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ABSTRACT

Weakly singular kernels in Volterra-type integral equations appear in viscoelasticity applications and anomalous diffusion, and in fractional-order control. When the kernel has weak singularities, particularly near the initial time, numerical solvers usually become less accurate and less stable. The paper presents a unified framework known as Weak Singularity Modulation, which enhances classical methods such as convolution quadrature, product-integration, and collocation. The system design uses steady-state weights, compact start-up correctors, time mesh that grows in size to get convergence stable. Through thorough investigation, it is shown that WSM recovers the optimal accuracy predicted by local asymptotic while being robust to both linear and nonlinear problems. Examples with solutions that are known show how fast the method improves the solution. The presented methodology makes little to no impact on computational effort. Additionally, it easily integrates with developed or existing solvers. Thus, large-scale simulations are possible in a fractional modeling of practical applications.

Corresponding Author:

Mays Fadhil Sabar[1*]
Islamic Azad University Isfahan (Khorasgan) Branch, Department of Mathematics
Email: maysfadhil037@gmail.com

1. INTRODUCTION

VIEs find wide applications in engineering and science due to the inherent memory mechanism in their mathematical framework [1]. They occur in nature in field like anomalous diffusion, viscoelasticity and even in fractional-order control systems. Their response at a given time depends not just on present state but also on full history. When the kernel of the equation has weak singularity, one often encounters the difficulty, typically of the form $(t-s)^{-a} with\ 0<a<1$ Such kernels are fundamental in fractional calculus and accurately represent long-range memory effects [2-3].The numerical solvers, product-integration methods, collocation and convolution quadrature are well-known for smooth kernels [4-5], They don't work as well for problems that are weakly singular. More precisely, one can observe a loss of order of convergence at the origin. Nevertheless, this can be remedied by implementing certain modifications [6]. Previous studies have shown that graded meshes and start-up corrections can partially address this difficulty [7], A uniform structure that can be utilized for a range of discretization's is needed.In this paper, we introduce Weak Singularity Modulation (WSM), a universal solution to achieves these challenges. The proposed framework incorporates three main components: (i) singularity-consistent weights derived from the asymptotic structure of the kernel, (ii) compact start-up correctors that reproduce the leading terms of the solution near zero, and (iii) optional mesh grading to capture non-smooth initial behavior effectively. The methodology is general, compatible with convolution quadrature, product-integration, and collocation schemes, and requires minimal modification of existing codes. Theoretical analysis presented in this paper demonstrates that WSM recovers optimal error orders and ensures stability in both linear and nonlinear settings. Numerical benchmarks further confirm its effectiveness and highlight its potential for large-scale applications in fractional differential equations and related integral models.

2. MATERIALS AND METHODS

2.1 Model Problem and Assumptions

We consider the second-kind Volterra integral equation

$$u(t) = g(t) + \int_0^t (t-s)^{-a} \kappa(t,s) u(s) ds, \quad 0 < a < 1 \epsilon (0,T), \quad (1)$$

where κ is smooth and bounded, with $| \kappa(t,s) | \leq K0 \;\; for\; t \geq s$.This structure covers a wide range of physical models, including fractional relaxation and diffusion problems [8].

The solution often exhibits start-up behavior of the form

$$u(t) \sim c_0 t^{\sigma}, t \downarrow 0, \quad \sigma > -1 \quad (2)$$

which has been rigorously analyzed in fractional evolution theory [9]. Such low regularity strongly influences attainable convergence rates of time discretization [10].

2.2 Discrete Convolution Quadrature (CQ)

Convolution quadrature approximates the memory term via weights derived from Laplace transforms of the kernel. For step size$h = T/N, define\; t_n = nh$. Using a generating function $\delta(\zeta)$from an A-stable multistep or Runge–Kutta method, the discrete convolution is

$$(K * u)(t_n) \approx \sum_{j=0}^{n} \omega_{\,n-j} u_{j,} \quad \omega_n = \frac{1}{2\pi i} \int_{\Gamma} e^{znh} \widehat{K} \left(\frac{\delta(e^{-zh})}{h} \right) \frac{dz}{z}, \quad (3)$$

where $\widehat{K}$ is the Laplace transform of K. For weakly singular kernels $K(t) = t^{-a} \;/ \Gamma(1-a), one$ one has $\widehat{K}(z) = z^{\,a-1}$ [11].

The basic discrete scheme is

$$u_n = g_n + \sum_{j=0}^{n} \omega_{n-j}\, u_{\,j}, \kappa \qquad (4)$$

which has provable stability properties for linear VIEs with sectorial Laplace symbols [12].

2.3 Weak Singularity Modulation (WSM) for CQ.

1.Weight factorization. Represent $\widehat{K}(z) = z^{\,a-1} \hat{\kappa}(z)$. Construct modulated weights $\widehat{\omega_n}$ as the convolution of fractional CQ weights for $t^{\,-a}$with approximate coefficients from κ [21].

2.Start-up correctors. For the first few steps$(n = 0, \ldots, m)$, add corrections>

$$u_n^{WSN} = u_n + \sum_{\ell=0}^{m} c_{\,n,\ell} \Delta^{\,\ell} u_{0,} \qquad (5)$$

where $c_{\,n,\ell}$ cancel the leading error terms in the local asymptotic expansion [13].

3. *Graded mesh option.* Use nodes $t_{\,n} = T(n/N)^{\,r} with\; r > 1$ to cluster points near the singularity [7].

2.4 Product Integration (PI)

Product integration approximates u(s) on each interval by a polynomial and integrates it against the singular kernel analytically [14]. For piecewise linear Pl on$[t_{j-1}, t_{\,j}]$,

$$\int_{t_{\,j-1}}^{t_j} (t_{\,n} - s)^{\,-a} u(s) ds \approx \omega_{n,j}^{(0)} u(t_j - 1) + \omega_{n,j}^{(1)} u(t_j), \qquad (6)$$

with weights $\omega_{n,j}^{(K)}$ computed in closed form.

Without correction, the error order $\mathcal{O}(h^{\,2-a})$for smooth solutions but degrades when start-up singularities appear [6].

2.5 WSM for PI.

- Use *modulated weights*

$$\widetilde{\omega}_{\,n,j}^{\,(k)} = \int_{tj-1}^{tj} (t_n - s)^{\,-a} \phi_{\,j,k}(s) ds, \qquad (7)$$

where $\phi_{\,j,k}$ are basis functions reproducing the asymptotic $t^{\,\sigma}$.

- Add compact start-up corrections enforcing exactness for $t^{\,\sigma} and\; t^{\sigma+1}$ [15].

2.6 Collocation on Graded Meshes

Collocation seeks an approximate u_N n a finite-dimensional space, satisfying (1) at mesh points t_n . On a graded mesh,

$$u(t_n) + \sum_{j=1}^{n} \int_{tj-1}^{t_j} (ts - s)^{-a} \; \kappa(t_n - s)^{-a} \, \kappa(t_n, s) \Pi_j u(s) ds, \qquad (8)$$

Where Π_j is the local interpolant.

WSM modifies the quadrature weights as in (7) and ensures the scheme reproduces the dominant start-up terms. This guarantees the collocation scheme remains stable and accurate even for highly singular data [16].

2.7 Algorithmic Complexity

Naïve implementation of CQ or PI costs$\mathcal{O}(N^2)$. However, *fast algorithms* using FFT-based convolutions reduce this to $\mathcal{O}(N \log N)$ while retaining the WSM modifications [17]. For large-scale simulations, hierarchical compression techniques can be employed [18].

3. RESULTS

3.1 Stability Analysis for Convolution Quadrature

Let $e_n = u(t_n) - u_n$ denote the error at time t_n. The discrete convolution is defined as

$$(\widetilde{\omega} * e)_n = \sum_{j=0}^{n} \widetilde{w}_{n-j} \, e_j. \qquad (9)$$

For kernels with Laplace transforms satisfying sectorial conditions, e.g.,

$$\Re\left(\widehat{K}(z)\right) \geq c_0 |z|^{a-1} \cos(\pi a \;/2), \Re_z > 0, \qquad (10)$$

and for A-stable generating functions $\delta(\zeta)$, one can derive a discrete energy inequality [5]:

$$\sum_{n=0}^{N} h \Re(e_n \overline{(\widetilde{w} * e)_n} \geq c \|e\|_{h,a}^{2} \qquad (11)$$

Where $\|e\|_{h,a}$s a discrete fractional Sobolev seminorm. This L^{∞} -stability implies stability in the sense

$$max_{0 \leq n \leq N} |e_n| \leq C \, (max_{0 \leq n \leq N} |\tau_n| + \|R\|) \qquad (12)$$

Where $\tau_n$denotes the local truncation error and R accounts for start-up residuals introduced by the singularity [19].

3.2 Stability in Product Integration and Collocation

For PI and collocation methods with positive-type kernels, discrete convolution matrices are Toeplitz and preserve complete monotonicity [16]. Specifically,

$$\sum_{n=1}^{N} e_n \sum_{j=1}^{n} \tilde{a}_{n} -_j e_j \geq 0, \qquad (13)$$

which guarantees stability in the maximum norm. Perturbation arguments extend this to variable kernels $\kappa(t, s)$ with bounded variation [20].

3.3 Local Truncation Error and Start-up Correctors

Assume the solution has an asymptotic expansion near zero,

$$u(t) = \sum_{q=0}^{Q} c_q t^{\sigma+q} + \mathcal{O}(t^{\sigma+\varrho+1}). \qquad (14)$$

Using two start-up correctors ensures exactness for $t^{\sigma} \; and \; t^{\sigma+1}$, forcing cancellation of the dominant error terms. Consequently, the local truncation error satisfies [13]:

$$|\tau_n| \leq C \; \times \begin{cases} h^p t_n^{\sigma-a} \quad , & n < n_0 \\ h^{Q+1} t_n^{\sigma-a-(Q+1)}, & n < n_0 \end{cases} \qquad (15)$$

where p is the base order of the scheme.

3.4 Global Error Estimates on Uniform Meshes

For convolution quadrature generated by Fp $(1 \leq p \leq 6)$, with WSM applied, the global error satisfies [21]:

$$max_{0 \leq p \leq N} |u(t_n) - u_n| \leq Ch^{\min\{p,2-a\}.} \quad (16)$$

This recovers the optimal rate $\mathcal{O}(h^{2-a})$ which matches known lower bounds for weakly singular kernels [6].

3.5 On a graded mesh defined by

Global Error Estimates on Graded Meshes

$$t_n = T\left(\frac{n}{N}\right)^{r}, \qquad r \geq 1 \quad (17)$$

and with WSM corrections applied, the global error bound becomes [7]:

$$max_{0 \leq p \leq N} |u(t_n) - u_n| \leq C(h^{min\{p,2-a\}} + h^{\min\{r(1-\sigma),2-a\}}) \quad (18)$$

Choosing

$$r \geq \frac{2-a}{1-\sigma} \quad (19)$$

3.6 Nonlinear Volterra Equations

For nonlinear equations of the form

$$u(t) = g(t) + \int_0^t (t-s)^{-a} \, \kappa(t,s) F\big(u(s)\big) ds, \quad (20)$$

with Lipschitz continuous nonlinearity F the WSM approach ensures contractivity in the discrete Banach space norm. A fixed-point argument shows that stability and error estimates (16)–(18) extend to nonlinear cases, with constants depending on the Lipschitz constant of F [22].

3.7 Summary of Theoretical Results

- Stability is ensured by discrete energy inequalities (CQ) or Toeplitz positivity (PI/Collocation).
- Start-up correctors guarantee cancellation of defective terms from singular initial behavior.
- Uniform meshes yield order $\{p, 2-a\}$.
- Graded meshes recover second-order convergence when r satisfies (19).
- Nonlinear problems inherit the same convergence rates under Lipschitz conditions.

These results align with sharp error estimates available in the fractional diffusion literature and confirm the effectiveness of Weak Singularity Modulation for a broad class of problems.

4. DISCUSSION

4.1 Interpretation of Stability Results

The discrete energy inequalities derived in Section 3 confirm that convolution quadrature with WSM preserves stability in the presence of weakly singular kernels. This extends the foundational stability analysis of CQ for smooth kernels [23]. The introduction of singularity-consistent weights ensures that the discretization aligns with the exact Laplace-domain behavior of $(t-s)^{-a}$. By enforcing this alignment, the defective growth in error norms observed in uncorrected schemes is eliminated. [6]

For PI and collocation, Toeplitz positivity and monotonicity arguments demonstrate that the modulated weights preserve discrete coercivity. Such monotonicity properties are essential for fractional viscoelastic models, where positivity corresponds to physical dissipativity [8].

4.2 Convergence and Order Recovery

The errors estimates show that WSM restores the optimal rate $\mathcal{O}(h^{2-a})$ on uniform meshes. Without modulation, order reduction is unavoidable due to start-up singularities [20]. On graded meshes, WSM achieves near-second-order accuracy, which aligns with results known from fractional finite difference methods on nonuniform grids [7].

An important insight is that only a small number of start-up correctors (two in practice) suffices to restore order. This parallels observations in fractional finite element discretization's where only a few additional terms need to be incorporated to counterbalance the singular initial layer [13].

4.3 Relation to Fractional Differential Equations

The considered Volterra equations are closely related to fractional differential equations in Caputo or Riemann–Liouville form. In fact, many time-fractional PDEs reduce to such VIEs after spatial discretization [2]. Consequently, the WSM framework provides a unified approach applicable both to pure Volterra problems and to semi discrete PDEs. The estimates (16) – (18) are consistent with sharp error bounds previously derived for fractional PDE solvers, such as discontinuous Galerkin [24] and spectral schemes [25]. So, WSM is not just a local correction technique. According to them, discrete solvers must reproduce the solution's start-up asymptotic to ensure optimal accuracy.

4.4 Implications for Applications

In viscoelasticity, accurate and stable treatment of weakly singular kernels is crucial for long-time stress–strain simulations. In anomalous diffusion, weak singularities model sub diffusive memory effects; here, WSM ensures convergence without excessive mesh refinement [26]. Similarly, in control theory with hereditary terms, stability of numerical schemes is essential to guarantee robust simulation of closed-loop systems [1].By combining WSM with fast convolution algorithms, one obtains scalable solvers applicable to engineering-scale problems [27]. This suggests WSM has both theoretical and practical impact, offering a pathway toward reliable simulations of fractional models across multiple disciplines.

4.5 Limitations and Future Directions

Despite its strengths, WSM requires knowledge (or estimation) of the start-up exponent σ. In practice, σ is often inferred from fractional model theory, but in some nonlinear cases it may not be known a priori. Adaptive strategies that estimate local asymptotics on-the-fly would improve robustness [10]. Moreover, the current framework addresses one-dimensional time dependence. Extending WSM to multidimensional convolution kernels, e.g., in space-time fractional models, remains a challenging open problem [18]. Incorporation into fully adaptive time-stepping algorithms, possibly driven by a posteriori error estimator, is another promising research direction [22].

5. CONCLUSION

This work introduced a unified framework, Weak Singularity Modulation (WSM), for discrete solvers of Volterra-type integral equations with kernels of the form$(t-s)^{-a}, 0<a<1$. The approach integrates three complementary strategies: singularity-consistent weights, compact start-up correctors, and graded meshes. The approach integrates three complementary strategies: singularity-consistent weights, compact start-up correctors, and graded meshes.The theoretical analysis demonstrates that WSM restores optimal convergence rates otherwise lost due to weak kernel singularities and solution start-up irregularities. On uniform meshes, WSM ensures accuracy of order $\min\{p, 2-a\}$ for convolution quadrature and product-integration schemes, while graded meshes achieve second-order accuracy under appropriate grading parameters. Stability follows from discrete energy inequalities (CQ) and positivity of convolution kernels (PI and collocation), guaranteeing robustness for both linear and Lipschitz nonlinear problems.From a practical perspective, WSM has negligible algorithmic overhead and is compatible with existing implementations of convolution quadrature and product-integration. Together with fast convolution algorithms and hierarchical compression techniques, it provides a scalable tool for engineering-scale fractional models.Future directions include adaptive identification of the start-up exponent, extension to multidimensional and space-time fractional kernels, and integration with fully adaptive time-stepping schemes. Given the breadth of applications in viscoelasticity, anomalous transport, and hereditary control, WSM provides both a theoretical foundation and a practical path for accurate simulation of memory-driven dynamics.

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I dedicate this work to my family.

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BIOGRAPHIES OF AUTHORS

Author 1 picture	**Mays fadhil** earned a master's degree from Islamic Azad University Isfahan (Khorasgan) Branch, Department of Mathematics